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Flip-moves and graded associative algebras

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Abstract. The relation between discrete topological field theories on triangulations of two-dimensional manifolds and associative algebras was worked out recently. The starting point for this development was the graphical interpretation of the associativity as flip of triangles. We show that there is a more general relation between flip-moves with two n-gons and Z_{n-2}-graded associative algebras. A detailed examination shows that flip-invariant models on a lattice of n-gons can be constructed from Z_2- or Z_1-graded algebras, reducing in the second case to triangulations of the two-dimensional manifolds. Related problems occur naturally in three-dimensional topological lattice theories.

Various aspects of topological lattice theories have been considered in the last few years. In Regge's discretized approach to gravity, with varying edge lengths and fixed coordination numbers, the discretized version of the two-dimensional Einstein action is topological due to the discretized Gauß–Bonnet-theorem [1, 2]. Other models with fixed edge length and varying lattice have been constructed as discrete analogues of continuous topological field theories. The invariance of the continuous theory under the diffeomorphism group is discretized to the invariance under flip-moves of the lattice [3], see figure 1. The field variables are located on the vertices of the triangulation. Another type of models arises from matrix models of two-dimensional quantum gravity [4], where one wants to couple a topological action to the model to control the topology dependence of the series expansion. In these models the field variables are defined on the edges of the triangles. They are classified by associative algebras [5, 6]. The approach to topological lattice theories from the matrix models poses the problem to handle 'topological' actions coupled to models which not only contain a cubic but higher monomials in the potential. This was solved in [6] for monomials of degree 4 and for arbitrary polynomials containing a cubic term, leading to quadrangulations of two-dimensional manifolds and to lattices built out of triangles and higher polygons.

This paper, which is based on [7], treats the remaining models for monomials of arbitrary degree, i.e. for manifolds covered by n-gons, $n \geqslant 3$. This is of interest not only in the two-dimensional case: it has been shown in [8] that for certain three-dimensional topological lattice theories one has to deal with polygons and multivalent hinges. They briefly discuss two-dimensional lattice theories with non-triangular faces and assume that the weights on the polygons have to be subdivision invariant. This is a result which in our work is a consequence of a rather natural condition on the weights.

From the given data we construct the sets of weights $\Gamma_{i_1 \ldots i_n}$ on the n-gons and the weights q^{ij} on the edges satisfying a flip condition, an associative graded algebra, which

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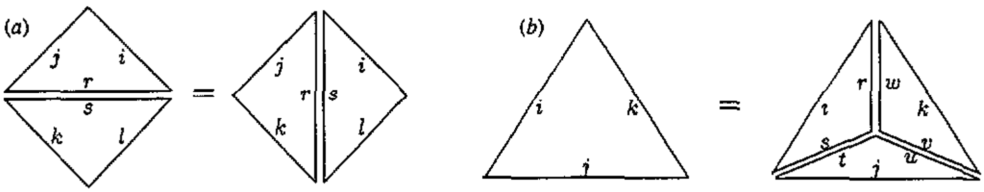

$$\Gamma_{ijr}q^{rs}\Gamma_{skl} = \Gamma_{jkr}q^{rs}\Gamma_{sli} \qquad \Gamma_{ijk} = \Gamma_{isr}q^{st}\Gamma_{jut}q^{uv}\Gamma_{kwv}q^{wr}$$

Figure 1. Moves for $n = 3$. (a) Flip-move and (b) pyramid move.

allows the classification of the models and the complete computation of the partition function. We recover two types of flip-invariant models: topological models on triangulations and models on chequered graphs, both already noted in [6]. This is a complete classification: all flip-invariant models on polygonizations belong to one of these two types. The effect of imposing a condition similar to the pyramid move is discussed at the end of this paper.

Let us now formalize the model. We consider a polygonization of a two-dimensional compact oriented manifold by n-gons. On this polygonization we establish a statistical model with variables $i, j, \ldots = 1, \ldots, N$ on the edges of the n-gons, weights $\Gamma_{i_1 \ldots i_n} \in \mathbb{C}$ on the n-gons and $q^{ij} \in \mathbb{C}$ on the edges. The weights Γ have to be cyclic, the weights q have to be symmetric.

We assume that the matrix (q^{ij}) is regular and the inverse matrix (q_{ij}) exists (this condition can always be achieved by a simple transformation and a reduction of the range of the indices, see [6]). The partition function is the sum of the product of all weights over all indices.

For $n = 3$ the model is called topological if the weights are invariant under the moves in figure 1, these moves act transitively on the set of all two-dimensional simplicial complexes with fixed Euler characteristic. This was already shown by Alexander in 1930 [9], see also [10] for a discussion. It is remarkable that on the set of regular graphs of degree 3 with genus h (i.e. graphs with vertices of valence 3 embedded in a two-dimensional manifold with h handles) and fixed number of vertices the dual move to the flip-move acts transitively. These graphs may contain double lines and self loops and therefore are not necessarily dual graphs of triangulations of the manifold. Actually in the proof [7] such graphs are used as representatives in each class.

In this case one defines an algebra $\mathcal{A}$ which is a vector space with basis $\{e_1, \ldots, e_N\}$ and multiplication $e_i \cdot e_j = \lambda_{ij}^k e_k$, where the structure constants are formed by $\lambda_{ij}^k = \Gamma_{ijr}q^{rk}$. (Here and in the following, sum convention is assumed.) The elements of the matrix (q_{ij}), the inverse matrix of (q^{ij}), form the coefficients of a symmetric bilinear form q on $\mathcal{A}$ with $q_{ij} = q(e_i, e_j)$. Due to the cyclicity of Γ_{ijk} this bilinear form is invariant under multiplication in $\mathcal{A}$:

$$q(a \cdot b, c) = q(a, b \cdot c) \qquad \forall\, a, b, c \in \mathcal{A} \tag{1}$$

An algebra together with a metric which fulfils (1) is called *metrized*, see e.g. [11].

As shown in several publications [5, 6], the conditions imposed by the flip and the pyramid move lead to an associative and semisimple algebra. The flip condition is the cause for associativity. The pyramid condition was thought to be the origin of the semisimplicity, but in [6] it was shown that the 'non-semisimple parts' of the algebra (which consists not only of the radical, but also of some Levi subalgebra) give no contributions to the partition function of the statistical models considered here, and can therefore be ignored. What

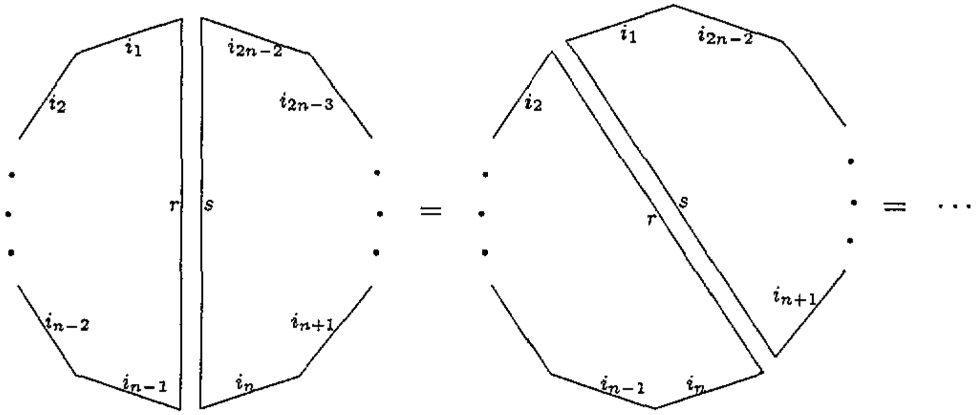

Figure 2. Flip-moves for two n-gons.

remains is a semisimple algebra. Imposing the pyramid move is therefore not necessary for the classification of topological models.

The relation between flip-moves and associative algebras was (see [6]) extended to the case of flips of two 4-gons, leading to $\mathbf{Z}_2$-graded associative algebras. There occurred the new feature that some of the models vanish on graphs which cannot be chequered.

We now generalize the work in [6] to arbitrary n-gons. First, we generalize the flip-move in figure 1 for two n-gons as shown in figure 2. Imposing a condition similar to the pyramid move will not be necessary for the classification of the topological models, see the concluding remarks.

The weights invariant under the moves in figure 2 fulfil the relations

$$\Gamma_{i_1\dots i_{n-1}r}q^{rs}\Gamma_{si_n\dots i_{2n-2}} = \Gamma_{i_2\dots i_n r}q^{rs}\Gamma_{si_{n+1}\dots i_{2n-2}i_1} = \dots .$$

Trivial examples of weights invariant under these flips are constructed from models on triangulations, the weight of the n-gon is defined by the fusion of the weights of $n-2$ triangles. A non-trivial example is the four-vertex model which was discussed in [4]. We will see that all models are similar to one of these examples.

As in the case $n = 3$ we define a N-dimensional complex vector space $\mathcal{A}$ with basis $\{e_1, \dots, e_N\}$ and a metric q on $\mathcal{A}$ by $q(e_i, e_j) = q_{ij}$. We define a $(n-1)$-linear map $\Gamma : \mathcal{A} \times \dots \times \mathcal{A} \mapsto \mathcal{A}$ by

$$\Gamma(e_{i_1}, \dots, e_{i_{n-1}}) := \Gamma_{i_1\dots i_{n-1}r}q^{rs}e_s .$$

Again the metric is invariant with respect to the map Γ:

$$\begin{aligned} q(\Gamma(e_{i_1}, \dots, e_{i_{n-1}}), e_{i_n}) &= \Gamma_{i_1\dots i_n} = \Gamma_{i_2\dots i_n i_1} \\ &= q(\Gamma(e_{i_2}, \dots, e_{i_n}), e_{i_1}) \\ &= q(e_{i_1}, \Gamma(e_{i_2}, \dots, e_{i_n})) . \end{aligned} \tag{2}$$

The flip condition in figure 2 imposes the following conditions on the map Γ:

$$\begin{aligned} \Gamma(\Gamma(a_1, \dots, a_{n-1}), a_n, \dots, a_{2n-3}) &= \Gamma(a_1, \Gamma(a_2, \dots, a_n), a_{n+1}, \dots, a_{2n-3}) \\ &= \dots = \Gamma(a_1, \dots, a_{n-2}, \Gamma(a_{n-1}, \dots, a_{2n-3})) \end{aligned} \tag{3}$$

which are equivalent to

$$\Gamma \circ (\mathrm{id}^r \otimes \Gamma \otimes \mathrm{id}^{n-2-r}) = \Gamma \circ (\mathrm{id}^s \otimes \Gamma \otimes \mathrm{id}^{n-2-s}) \tag{4}$$

for all $r, s = 0, \ldots, n-2$. This is a generalization of the associativity condition of associative algebras. An easy but time-consuming induction shows that this general associativity holds for more than two Γ:

$$\Gamma \circ (\mathrm{id}^{r_1} \otimes \Gamma \otimes \mathrm{id}^{n-2-r_1}) \circ \cdots \circ (\mathrm{id}^{r_k} \otimes \Gamma \otimes \mathrm{id}^{k(n-2)-r_k})$$
$$= \Gamma \circ (\mathrm{id}^{s_1} \otimes \Gamma \otimes \mathrm{id}^{n-2-s_1}) \circ \ldots \circ (\mathrm{id}^{s_k} \otimes \Gamma \otimes \mathrm{id}^{k(n-2)-s_k}) \tag{5}$$

for all admissible r_i and s_i.

For practical reasons we rename the vector space $\mathcal{A}$ by $\mathcal{A}_1$ and the metric q by q_1.

Then we can prove the following main theorem:

Theorem 1. Let $\mathcal{A}_1$ be a N-dimensional complex vector space. Let $\Gamma : \mathcal{A}_1^{\times n-1} \to \mathcal{A}_1$ be a $\mathbb{C}$-multilinear map and $q_1 : \mathcal{A}_1 \times \mathcal{A}_1 \to \mathbb{C}$ a symmetric, non-degenerate metric, which satisfy the invariance condition (2) and the general associativity condition (3).

Then there exists a $\mathbf{Z}_{n-2}$-graded, associative, metrized algebra $(\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_{n-3}, q)$, where q is a non degenerate, symmetric bilinear form on $\mathcal{A}$ with $q|_{\mathcal{A}_1 \times \mathcal{A}_1} = q_1$ and $q|_{\mathcal{A}_i \times \mathcal{A}_j} = 0$ for $i + j \not\equiv 2 \mod (n-2)$.

The map Γ and the algebra multiplication are related by

$$\Gamma(a_1, \ldots, a_{n-1}) = a_1 \cdot \cdots \cdot a_{n-1} \qquad \forall\, a_1, \ldots, a_{n-1} \in \mathcal{A}_1 . \tag{6}$$

Remark. An algebra $\mathcal{A} = \oplus_{i=0}^{m-1} \mathcal{A}_i$ is $\mathbf{Z}_m$-graded, if the multiplication fulfils $\mathcal{A}_i \times \mathcal{A}_j \to \mathcal{A}_{i+j \bmod m}$. Hence $a_1 \cdot a_2 \in \mathcal{A}_2$, $a_1 \cdot a_2 \cdot a_3 \in \mathcal{A}_3 \ldots$, and finally $a_1 \ldots a_{n-1} \in \mathcal{A}_1$. We remark, that the algebra is not supergraded, as it is assumed automatically by Lie algebras.

Let $\{e_1, \ldots, e_{|\mathcal{A}|}\}$ be an ordered basis of $\mathcal{A}$ with respect to the grading. Let λ_{ij}^k be the structure constants with respect to this basis. Then we get with (6),

$$\begin{aligned}\Gamma_{i_1 \ldots i_n} &= q_1(\Gamma(e_{i_1}, \ldots, e_{i_{n-1}}), e_{i_n}) = q(e_{i_1} \ldots e_{i_{n-1}}, e_{i_n}) \\ &= q_{r i_n} \lambda_{i_1 i_2}^{r_1} \lambda_{r_1 i_3}^{r_2} \ldots \lambda_{r_{n-3} i_{n-1}}^{r} \\ &= \lambda_{i_1 i_2 r_1} q^{r_1 s_1} \lambda_{s_1 i_3 r_2} q^{r_2 s_2} \ldots q^{r_{n-2} s_{n-2}} \lambda_{s_{n-2} i_{n-1} i_n} .\end{aligned} \tag{7}$$

The inner indices are summed over $1, \ldots, \dim \mathcal{A}$, but due to the grading of the algebra every summation is restricted to the indices belonging to one part of the grading.

Thanks to the associativity we can replace the right-hand side of (7) by any evaluation of the associative product in $q(e_{i_1} \ldots e_{i_{n-1}}, e_{i_n})$. The graphical interpretation is simple: we can replace the n-gon with weight $\Gamma_{i_1 \ldots i_n}$ by a triangulation with $n-2$ triangles and weights λ_{ijk} and summation over all inner indices, as in figure 3. Due to the associativity of the algebra $\mathcal{A}$ this model flip is invariant. We see flip-invariant models on n-gonizations of a two-dimensional manifold are equivalent to flip-invariant models on triangulations with a greater range of indices and the restriction that certain indices only take values in the original part. The value $n-2$ will appear often, therefore we define $p := n-2$.

To prove the theorem we have to perform the following steps:

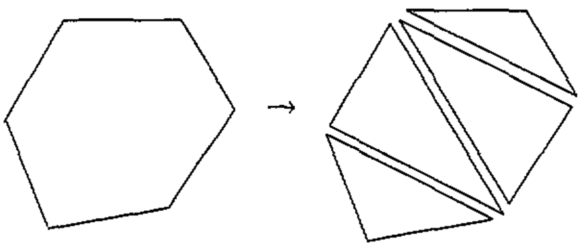

Figure 3. Split n-gon.

(i) We define a non-associative graded algebra structure on the vector space $M = \oplus_{k=1}^{p} \mathcal{A}_1^{\otimes k}$ by the multiplication

$$a \cdot b = \begin{cases} a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_l = a \otimes b & k+l \leqslant p \\ \Gamma(a_1, \cdots, b_{p+1-k}) \otimes b_{p+2-k} \otimes \cdots \otimes b_l & k+l > p \end{cases} \tag{8}$$

for $a = a_1 \otimes \cdots \otimes a_k$ and $b = b_1 \otimes \cdots \otimes b_l$. This algebra is finite-dimensional, but non-associative. The properties of Γ allow the definition of an ideal I, such that M/I is associative. This is not the usual way to construct an associative algebra, which would start with the infinite dimensional universal tensor algebra over $\mathcal{A}_1$ and divide out an infinite-dimensional ideal to get a finite-dimensional associative algebra.

(ii) We define the subspace $I := \oplus_{k=1}^{p} I_k$ of M with $I_1 := \{0\}$ and

$$I_k \ := \ \{a \in \mathcal{A}_1^{\otimes k} | a \cdot b = 0 \ \forall \ b \in \mathcal{A}_1^{p+1-k}\} \tag{9}$$

for $k = 2, \ldots, p$ and show that I is a two-sided ideal of M.

(iii) We can therefore define the algebra

$$\mathcal{A} \ := \ M/I = \bigoplus_{k=1}^{p} \mathcal{A}_1^{\otimes k}/I_k \ =: \ \bigoplus_{k=1}^{p} \mathcal{A}_k \tag{10}$$

which will be shown to be associative and contains the original vector space $\mathcal{A}_1$. The relation (6) concides with the definition of the multiplication.

(iv) We define a bilinear form q on M for $a = a_1 \otimes \cdots \otimes a_k, b = b_1 \otimes \cdots \otimes b_l \in M$ by

$$q(a,b) := \begin{cases} 0 & k+l \not\equiv 2 \mod p \\ q_1(a_1, \Gamma(a_2, \ldots, b_l)) & k+l = n \\ q_1(a_1, b_1) & k = l = 1 \end{cases} \tag{11}$$

and show that the projection of q on $\mathcal{A}$, which we will also denote by q, is well defined and symmetric.

(v) We show that q is non-degenerate on $\mathcal{A}$.

(vi) We show that q is invariant on $\mathcal{A}$.

Proofs and Remarks.

(i) The multiplication (8) is, in general, not associative, consider, for example,

$$\begin{aligned} a \cdot (b_1 \otimes \cdots \otimes b_p \cdot c) &= a \otimes \Gamma(b_1, \ldots, b_p, c) \\ (a \cdot b_1 \otimes \cdots \otimes b_p) \cdot c &= \Gamma(a, b_1, \ldots, b_p) \otimes c \ . \end{aligned}$$

But we can show that the multiplication is associative for factors $a \in \mathcal{A}_1^{\otimes n_a}$, $b \in \mathcal{A}_1^{\otimes n_b}$, $c \in \mathcal{A}_1^{\otimes n_c}, \ldots$ with $n_a + n_b + n_c + \cdots \equiv 1 \bmod p$, e.g.

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{12}$$

$$((a \cdot b) \cdot c) \cdot d = (a \cdot (b \cdot c)) \cdot d \, . \tag{13}$$

For this, let $a = a_1 \otimes \cdots \otimes a_{n_a}$, $b_. = b_1 \otimes \cdots \otimes b_{n_b}$, $c = c_1 \otimes \cdots \otimes c_{n_c}, \ldots$. Since $n_a + n_b + n_c + \cdots \equiv 1 \bmod p$, all the products are of the form

$$\Gamma \circ (\mathrm{id}^{r_1} \otimes \Gamma \otimes \mathrm{id}^{p-r_1}) \circ (\mathrm{id}^{r_2} \otimes \Gamma \otimes \mathrm{id}^{2p-r_2}) \circ \ldots (a_1, \ldots, a_{n_a}, b_1, \ldots)$$

and products of the same factors are equivalent by (5).

(ii) The condition $a \cdot b = 0$ for all $b \in \mathcal{A}_1^{\otimes p+1-k}$ is equivalent to $b \cdot a = 0$ for all $b \in \mathcal{A}_1^{\otimes p+1-k}$ and we can define alternatively

$$I_k = \{a \in \mathcal{A}_1^{\otimes k} | b \cdot a = 0 \ \forall \ b \in \mathcal{A}_1^{\otimes p+1-k}\} \, . \tag{14}$$

In order to see this, we use the invariance condition (2) and the symmetry of q_1: For all $b = b_1 \otimes \ldots \otimes b_{p+1-k} \in \mathcal{A}_1^{\otimes p+1-k}$, for all $c \in \mathcal{A}_1$ and for $a = a^{i_1 \ldots i_k} e_{i_1} \otimes \cdots \otimes e_{i_k} \in I_k$ holds

$$\begin{aligned}
& 0 = a \cdot b = a^{i_1 \ldots i_k} \Gamma(e_{i_1}, \ldots, e_{i_k}, b_1, \ldots, b_{p+1-k}) \\
\Leftrightarrow\ & 0 = q_1(a^{i_1 \ldots i_k} \Gamma(e_{i_1}, \ldots, e_{i_k}, b_1, \ldots, b_{p+1-k}), c) \\
\Leftrightarrow\ & 0 \overset{(2)}{=} q_1(a^{i_1 \ldots i_k} \Gamma(b_2, \ldots, b_{p+1-k}, c, e_{i_1}, \ldots, e_{i_k}), b_1) \\
\Leftrightarrow\ & 0 = a^{i_1 \ldots i_k} \Gamma(b_2, \ldots, b_{p+1-k}, c, e_{i_1}, \ldots, e_{i_k}) \\
\Leftrightarrow\ & 0 = (b_1 \otimes \cdots \otimes b_{p+1-k} \otimes c) \cdot a\,.
\end{aligned}$$

Since the elements of the form $b_1 \otimes \cdots \otimes b_{p+1-k} \otimes c$ span $\mathcal{A}_1^{\otimes p+1-k}$, the equivalence is proved.

The subspace I of M is an ideal of M: Let $a \in I_{n_a}$, then for all $b \in \mathcal{A}_1^{\otimes n_b}$, $n_b = 1, \ldots, p$ and all $c \in \mathcal{A}_1^{\otimes n_c}$ with n_c such that $n_a + n_b + n_c \equiv 1 \bmod p$, we have

$$(a \cdot b) \cdot c \overset{(12)}{=} a \cdot (b \cdot c) = 0 \quad \Rightarrow a \cdot b \in I \tag{15}$$

since $b \cdot c \in \mathcal{A}_1^{\otimes p+1-k}$ and

$$(b \cdot a) \cdot c \overset{(12)}{=} b \cdot \underbrace{(a \cdot c)}_{\in I \text{ by } (15)} = 0$$

by (14), hence also $b \cdot a \in I$ and I is a two sided ideal.

(iii) The algebra defined in (10) is associative. To prove this, we have to show that for $a, b, c \in M$ holds $(a \cdot b) \cdot c - a \cdot (b \cdot c) \in I$. Again it is sufficient to consider $a \in \mathcal{A}_1^{\otimes n_a}$, $b \in \mathcal{A}_1^{\otimes n_b}$, $c \in \mathcal{A}_1^{\otimes n_c}$. Let $d \in \mathcal{A}_1^{\otimes n_d}$ with n_d such that $n_a + n_b + n_c + n_d \equiv 1 \bmod p$. Then

$$\begin{aligned}
((a \cdot b) \cdot c) \cdot d &\overset{(13)}{=} (a \cdot (b \cdot c)) \cdot d \\
&\Rightarrow ((a \cdot b) \cdot c - a \cdot (b \cdot c)) \cdot d = 0
\end{aligned}$$

and since this holds for all d, the difference is in I and the algebra $\mathcal{A}$ is associative.

We define $\mathcal{A}_0 := \mathcal{A}_p$. With the multiplication $\mathcal{A}_i \cdot \mathcal{A}_j \rightarrow \mathcal{A}_{i+j \bmod p}$ becomes $\mathcal{A} = \mathcal{A}_0 \oplus \ldots \oplus \mathcal{A}_{n-3}$ a $\mathbf{Z}_p$-graded associative algebra. The condition (6) is satisfied due to the definition (8): Let $a_1, \ldots, a_{p+1} \in \mathcal{A}_1$. Then

$$a_1 \cdot a_{p+1} = \Gamma(a_1, \ldots, a_{p+1})\,.$$

(iv) The map $q : M \times M \rightarrow \mathbb{C}$ defined in (11) is well defined on $\mathcal{A} \times \mathcal{A}$. To prove this we have to show that $q(a, b)$ is independent of the choice of the representatives of a and b, i.e. for all $c_a, c_b \in I$ holds: $q(a + c_a, b + c_b) = q(a, b)$, i.e. $q(a, c_b) = 0 = q(c_a, b) = q(c_a, c_b)$. Due to the block structure of q it is sufficient to consider for a, b, c_a and c_b only homogeneous elements. Let $a = a_1 \otimes \cdots \otimes a_k \in \mathcal{A}_1^{\otimes k}$, $c_b = c^{j_1 \ldots j_l} e_{j_1} \otimes \ldots \otimes e_{j_l} \in I_l$, $k + l \equiv 2 \bmod p$. In the case $k = l = 1$ $c_b = 0$ and $q(a, c_b) = 0$, for $k + l = n$ we have

$$\begin{aligned}
q(a, c_b) &= q(a_1 \otimes \cdots \otimes a_k, c^{j_1 \ldots j_l} e_{j_1} \otimes \ldots \otimes e_{j_l}) \\
&= q_1(a_1, c^{j_1 \ldots j_l} \Gamma(a_2, \ldots, a_k, e_{j_1}, \ldots, e_{j_k})) \\
&= q_1(a_1, \underbrace{(a_2 \otimes \cdots \otimes a_k) \cdot c_b}_{=0}) = 0\,.
\end{aligned}$$

analogous $q(c_a, b) = 0$, $q(c_a, c_b) = 0$ is then clear.

q is symmetric: for $a = a_1 \otimes \cdots \otimes a_k$, $b = b_1 \otimes \cdots \otimes b_l$ is

- $k + l \not\equiv 2 \mod p : q(a, b) = 0 = q(b, a)$
- $k = l = 1 : q(a, b) = q_1(a, b) = q_1(b, a) = q(b, a)$
- $k + l = n :$

$$q(a, b) = q(a_1 \otimes \cdots \otimes a_k, b_1 \otimes \cdots \otimes b_l) = q_1(a_1, \Gamma(a_2, \ldots, a_k, b_1, \ldots, b_l))$$
$$\stackrel{(2)}{=} q_1(b_1, \Gamma(b_2, \ldots, b_l, a_1, \ldots, a_k)) = q(b, a) .$$

(v) q is non-degenerate. Due to the block structure of the metric it is again sufficient to consider homogeneous elements $a \in \mathcal{A}_1^{\otimes k}$ and $c = c^{i_1 \ldots i_l} e_{i_1} \otimes \cdots \otimes e_{i_l} \in \mathcal{A}_1^{\otimes l}$, $k + l \equiv 2 \mod p$. Let $q(a, c) = 0$ for all $a = a_1 \otimes \cdots \otimes a_k \in \mathcal{A}_1^{\otimes k}$:

$$\Leftrightarrow q(a_1 \otimes \cdots \otimes a_k, c^{j_1 \ldots j_l} e_{j_1} \otimes \ldots \otimes e_{j_l}) = 0 \;\; \forall\, a_1, \ldots a_k \in \mathcal{A}_1$$
$$\Leftrightarrow q_1(a_1, c^{j_1 \ldots j_l} \Gamma(a_2, \ldots, a_k, e_{j_1}, \ldots, e_{j_l})) = 0 \forall\, a_1, \ldots, a_k \in \mathcal{A}_1$$
$$\Leftrightarrow c^{j_1 \ldots j_l} \Gamma(a_2, \ldots, a_k, e_{j_1}, \ldots, e_{j_l}) = (a_2 \otimes \ldots \otimes a_k) \cdot c = 0$$
$$\Leftrightarrow c \in I_l .$$

(vi) To prove the invariance of the metric we consider again $a = a_1 \otimes \cdots \otimes a_{n_a}$, $b = b_1 \otimes \cdots \otimes b_{n_b}$, $c = c_1 \otimes \cdots \otimes c_{n_c}$. For $n_a + n_b + n_c \not\equiv 2 \mod p$ we have

$$q(a, b \cdot c) = 0 = q(a \cdot b, c) .$$

For $n_a + n_b + n_c = n$ we have

$$\begin{aligned} q(a \cdot b, c) &= q(c, a \cdot b) = q_1(c_1, \Gamma(c_2, \ldots, c_{n_c}, a_1, \ldots, a_{n_a}, b_1, \ldots, b_{n_b})) \\ &= q_1(a_1, \Gamma(a_2, \ldots, a_{n_a}, b_1, \ldots, b_{n_b}, c_1, \ldots, c_{n_c})) = q(a, b \cdot c) . \end{aligned} \tag{16}$$

For $n_a + n_b + n_c \equiv 2 \mod p$ let $a' = a_2 \otimes \cdots \otimes a_k$, i.e. $a = a_1 \cdot a'$. Then

$$q(a \cdot b, c) = q((a_1 \cdot a') \cdot b, c) = q(a_1 \cdot (a' \cdot b), c)$$
$$\stackrel{(16)}{=} q(a_1, (a' \cdot b) \cdot c) = q(a_1, a' \cdot (b \cdot c))$$
$$\stackrel{(16)}{=} q(a_1 \cdot a', b \cdot c) = q(a, b \cdot c) .$$

The metric q has a block structure with respect to the decomposition $\mathcal{A} = \oplus_{k=0}^{n-3} \mathcal{A}_k$, due to $q(\mathcal{A}_i, \mathcal{A}_j) = 0$ for $i + j \not\equiv 2 \mod p$ we get for $n > 4$

$$q = (q_{ij}) = \begin{pmatrix} 0 & & \square & & \\ & \square & & 0 & \\ \square & & & & \\ & & & & \square \\ & 0 & & \cdot^{\cdot^{\cdot}} & \\ & & \square & & 0 \end{pmatrix} . \tag{17}$$

The ith column and row, respectively, belong to the component $\mathcal{A}_{i-1}$ of $\mathcal{A}$. We use the symbol q for the metric and for the matrix (q_{ij}) in a basis. We assume in the following, that we have chosen a basis $\{e_i\}$ which respects the grading of $\mathcal{A}$. It is easy to see that the inverse matrix (q^{ij}) then has the same structure; matrix elements q^{ij} are only not equal to zero if the basis elements e_i and e_j lie in components $\mathcal{A}_k$ and $\mathcal{A}_l$ with $k + l \equiv 2 \mod p$. □

We now use the methods elaborated in [6] to calculate the partition functions of the flip-invariant models. We first review a few facts about associative, metrized algebras (see [6] for details).

- Let $\mathcal{A}$ be a complex, associative, metrized algebra. We decompose $\mathcal{A} = \mathcal{B} \oplus L \oplus R$, where $\mathcal{B}$ is the largest semisimple ideal of $\mathcal{A}$, L is a (non-unique) semisimple Levi-subalgebra and R is the radical of $\mathcal{A}$. $\mathcal{B}$ and $L \oplus R$ are orthogonal with respect to q, i.e. $L \oplus R = \mathcal{B}^{\perp}$.
- $q^{ij} \neq 0$ for $e_i \in L$ is only possible if $e_j \in R$
- $\mathcal{B}$ itself is the direct sum of the simple ideals of $\mathcal{A}$, $\mathcal{B} = \oplus_i I_i$, where I_i are the simple ideals of $\mathcal{A}$ and these are all orthogonal: $I_i \perp I_j$ for $i \neq j$.

We now check the relation of the decompositions $\mathcal{A} = \mathcal{B} \oplus L \oplus R$ and $\mathcal{A} = \oplus_k \mathcal{A}_k$. To this end we introduce the grading operator θ on $\mathcal{A}$ by $\theta(a_k) = \omega^k a_k$ for $a_k \in \mathcal{A}_k$, $\omega = \exp(2\pi \mathrm{i}/p)$. θ is an automorphism of $\mathcal{A}$ since $\mathcal{A}_k \times \mathcal{A}_l \to \mathcal{A}_{k+l \bmod p}$.

Every θ-invariant subalgebra X of $\mathcal{A}$ allows a decomposition $X = \oplus_k X_k$ with $X_k \subset \mathcal{A}_k$. $\mathcal{B}$ is a θ-invariant subalgebra, since the image of a semisimple ideal is a semisimple ideal, therefore $\mathcal{B} = \oplus_k \mathcal{B}_k$. By the theorem 1 in [12] there exists a θ-invariant Levi algebra L which allows a decomposition $L = \oplus_k L_k$. The image of the radical R is the radical, hence we also have $R = \oplus_k R_k$.

We therefore have a decomposition of each $\mathcal{A}_k = \mathcal{B}_k \oplus L_k \oplus R_k$ and we can choose a basis of $\mathcal{A}$ respecting this decomposition.

Since $\mathcal{B}$ and $L \oplus R$ are orthogonal the partition function splits into the partition function of a model with the semisimple algebra $\mathcal{B}$ and of the algebra $L \oplus R$. The latter can be shown to be zero, the arguments are the same as in [6], we will only give a sketch of the discussion.

We consider the split graph, let i_1 be an arbitrary index. Let $e_{i_1} \in R$, we consider all triangles which contain the vertex opposite to the index i_1. We label the indices as in figure 4. The partition function of this part of the graph, summed over all inner indices $r_1, \ldots, r_N$ and $s_1, \ldots, s_N$, is given by

$$\begin{aligned} Z_{i_1,\ldots,i_N} &= \lambda_{r_1 i_1 s_1} q^{s_1 r_2} \lambda_{r_2 i_2 s_2} \ldots \lambda_{r_N i_N s_N} q^{s_N r_1} = \lambda^{r_2}_{r_1 i_1} \lambda^{r_3}_{r_2 i_2} \ldots \lambda^{r_1}_{r_N i_N} \\ &= (R_{i_N} \cdot R_{i_{N_1}} \cdot \ldots \cdot R_{i_1} e_{r_1})^{r_1} = (R_{e_{i_1} e_{i_2} \ldots e_{i_N}} e_{r_1})^{r_1} \\ &= \operatorname{tr} R_{e_{i_1} e_{i_2} \ldots e_{i_N}} \end{aligned}$$

where R_a is the right multiplication in $\mathcal{A}$ considered as an endomorphism: $R_a b = ba$, R_i is short for R_{e_i}. If $e_{i_1} \in R$, then is also $e_{i_1} e_{i_2} \ldots e_{i_N} \in R$ and the trace vanishes, therefore all configurations with an index in R give no contribution to the partition function.

Now let $e_i \in L$, then $q^{ij} = 0$ for all $e_j \notin R$, but if $e_j \in R$ then we can repeat the discussion above with the result that also all configurations with an index in L give no

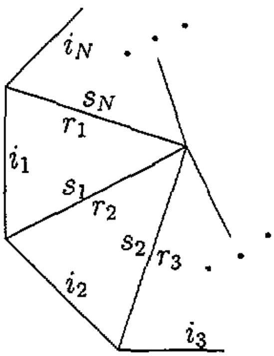

Figure 4. Indices on the split graph.

contribution to the partition function.

There remains the discussion of the semisimple algebra $\mathcal{B}$. It is θ-invariant and the (orthogonal) direct sum of all simple ideals of $\mathcal{A}$. One might expect that each simple ideal I is itself θ-invariant, but this is not true in general.

θ is an automorphism of $\mathcal{A}$, the image of a simple ideal I_1 is also a simple ideal $I_2 = \theta(I_1)$ which can be different from I_1. We get a sequence $I_1, I_2, \ldots, I_k$ of disjoint isomorphic simple ideals with $\theta(I_k) = I_1$; since $\theta^p = 1$ the number k must be a divisor of p, $p = kl$. Not each ideal I_i is θ-invariant, but the direct sum $I = I_1 \oplus \ldots \oplus I_k$ is, and we can decompose it in $I = I^{(0)} \oplus \ldots \oplus I^{(p-1)}$ with $I^{(j)} \subset \mathcal{A}_j$ and $\theta(I^{(j)}) = \omega^j I^{(j)}$. The partition function decomposes into several parts belonging to θ-invariant semisimple ideals of $\mathcal{A}$.

By the assumptions in theorem 1 $q|_{I^{(1)}}$ is non-degenerate. We will test this condition to gain information about k: let $a, b \in I^{(1)}$, $a = a_1 + \cdots + a_k$, $b = b_1 + \cdots + b_k$, $a_j, b_j \in I_j$. Since $\theta(a) = \omega a$ and $\theta(a_j) \in I_{j+1}$ we get $\theta(a_j) = \omega a_{j+1}$, $\theta(a_k) = \omega a_1$ and therefore

$$
\begin{aligned}
a_j = \omega^{1-j}\theta^{j-1}(a_1) \Rightarrow a &= \sum_j \omega^{1-j}\theta^{j-1}(a_1) \\
b &= \sum_j \omega^{1-j}\theta^{j-1}(b_1)
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow q(a,b) &= \sum_{i,j} \omega^{2-i-j} q(\theta^{i-1}(a_1), \theta^{j-1}(b_1)) = \sum_i \omega^{2(1-i)} q(\theta^{i-1}(a_1), \theta^{i-1}(b_1)) \\
&= \sum_i \omega^{2(1-i)}\omega^{2(i-1)} q(a_1, b_1) = kq(a_1, b_1)
\end{aligned}
$$

where we have used $q(\theta(a), \theta(b)) = \omega^2 q(a,b)$. θ^k is an automorphisms of I_1, which is a simple complex algebra isomorphic to a full complex matrix algebra. By the theorem of Noether–Skolem [13] is θ^k an inner automorphism, i.e. there exists an invertible element $s \in I_1$ with $\theta^k(a) = s^{-1}as$ for all $a \in I_1$. Then

$$
\begin{aligned}
q(\theta^k(a_1), \theta^k(b_1)) &= \omega^{2k} q(a_1, b_1) \\
&= q(s^{-1}a_1 s, s^{-1}b_1 s) = q(a_1, b_1) \qquad \forall a_1, b_1 \in I_1
\end{aligned}
$$

where we have used the invariance and the symmetry of q. Hence $\omega^{2k} = 1$ which is only possible for $2k = p$ or $k = p$. All other cases, e.g. $k = 1$ for $p > 2$ which corresponds to a θ-invariant simple ideal do not occur in the context of flip-invariant models.

There remains the discussion of these two cases:

$k = p$. This is the trivial one. There are p simple ideals isomorphic to a full complex matrix algebra $\mathbb{C}^{r\times r}$ Let $\{e_i\}$ be a basis of I_1, then is $\{\tilde{e}_i = e_i + \omega^{-1}\theta(e_i) + \cdots + \omega^{1-p}\theta^{p-1}(e_i)\}$ a basis of $\mathcal{A}_1$. Denote by $(a)_1$ the I_1 component of a, then we get for the weights

$$
\begin{aligned}
\Gamma_{i_1 \ldots i_n} &= q(\Gamma(\tilde{e}_{i_1}, \ldots, \tilde{e}_{i_{p+1}}), \tilde{e}_{i_n}) = q(\tilde{e}_{i_1} \ldots \tilde{e}_{i_{p+1}}, \tilde{e}_{i_n}) \\
&= kq^{(1)}((\tilde{e}_{i_1} \ldots \tilde{e}_{i_{p+1}})_1, (\tilde{e}_{i_n})_1) = kq^{(1)}(e_{i_1} \ldots e_{i_{p+1}}, e_{i_n}) \,. \qquad (18)
\end{aligned}
$$

This is exactly the weight one would get for a n-gon glued together out of $n-2$ triangles with a topological weight on the triangles. Therefore this case is called trivial.

For the calculation of the partition function it is convenient to consider the dual graph in the double-line representation [6]. We choose in I_1, which is isomorphic to a full complex matrix algebra, the standard basis $\{E_{ij}\}$ of $r \times r$ matrices with $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

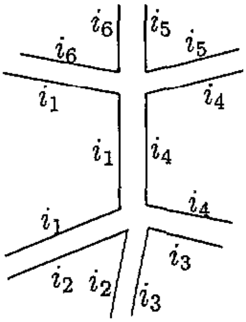

Figure 5. Double-line representation with equal indices.

$q^{(1)} = q|_{I_1\times I_1}$ is an invariant metric on I_1, this is, up to a factor, the trace of the matrices: $q^{(1)}(a,b) = \beta\,\mathrm{tr}(ab)$. We get for the weights of the vertices of degree n

$$\Gamma_{i_1 j_1 i_2 j_2 \ldots i_n j_n} = p\beta\delta_{j_1 i_2}\delta_{j_2 i_3}\ldots\delta_{j_n i_1} \tag{19}$$

and for the weights of the edges ($q(\tilde{a},\tilde{b}) = pq^{(1)}(a,b)$)

$$q^{i_1 j_1 i_2 j_2} = (p\beta)^{-1}\delta_{j_1 i_2}\delta_{j_2 i_1}\,. \tag{20}$$

All indices on a closed line must have the same value as indicated in figure 5, each closed line corresponds to a vertex of the original graph. The computation of the partition function is therefore reduced to a counting of factors. We get a factor r for each vertex of the polygonization, a factor $(p\beta)^{-1}$ for each edge and a factor $p\beta$ for each n-gon. This results in

$$Z = r^V(p\beta)^{-E}(p\beta)^P = (p\beta)^\chi\left(\frac{r}{p\beta}\right)^V \tag{21}$$

where V is the number of vertices of the polygonization, E is the number of edges and P the number of plaquettes, the n-gons. We get the typical dependence of the partition function on the Euler characteristic χ of the manifold. If one adjusts the constant β, such that $r = p\beta$, then the partition function will be topological. See [6] and the end of this text for a discussion.

The other case $2k = r$ is non-trivial and leads to new aspects. Let $a = a_1+\cdots+a_k \in I^{(1)}$, i.e. $\theta(a) = \omega a$. Then $a_i = \omega^{1-i}\theta^{i-1}(a_1)$, $\theta^k(a_1) = \omega^k a_1 = -a_1$. In this case is $\Theta = \theta^k$ is an automorphism from I_1 to I_1 with $\Theta^2 = 1$. By the theorem of Noether–Skolem it is an inner automorphism, there exists a $s \in I_1$ with $\Theta(a) = s^{-1}as$ for all $a \in I_1$. Then $\Theta^2(a) = s^{-2}as^2 = a$ i.e. $[a, s^2] = 0$ for all $a \in I_1$. With Schur's Lemma we conclude that $s^2 = \lambda 1$, we can set $\lambda = 1$. Then we can choose a basis in I_1 such that $s = \mathrm{diag}(1,\ldots,1,-1,\ldots,-1)$ with M times 1 and N times -1, $M + N = r$. $\Theta(a_1) = s^{-1}a_1 s = -a_1$ is fulfilled for all matrices $a_1 \in I_1$ which have the off-diagonal block form

$$a_1 = \begin{pmatrix} 0 & \square \\ \square & 0 \end{pmatrix} \tag{22}$$

i.e. $(a_1)_{ij} = 0$ for $i, j \leqslant M$ or for $i, j > M$. I_1 is then a $\mathbf{Z}_2$-graded algebra. A basis of $I^{(1)}$ is given by $\{\tilde{E}_{ij} = E_{ij} + \omega^{-1}\theta(E_{ij}) + \ldots + \omega^{1-k}\theta^{k-1}(E_{ij}) | i \leqslant M < j \text{ or } j \leqslant M < i\}$. The weights are given by

$$\Gamma_{i_1 j_1 i_2 j_2 \ldots i_n j_n} = k\beta\delta_{j_1 i_2}\delta_{j_2 i_3}\ldots\delta_{j_n i_1} \tag{23}$$

$$q^{i_1 j_1 i_2 j_2} = (k\beta)^{-1}\delta_{j_1 i_2}\delta_{j_2 i_1} \tag{24}$$

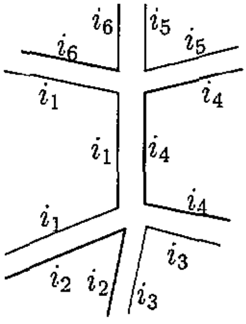

Figure 6. Double-line representation of a chequered graph

where the pairs $i_1 j_1, i_2 j_2, \ldots, i_n j_n$ fulfil the alternating relations $i \leqslant M < j$ and $j \leqslant M < i$.

In the double-line representation of the dual graph each double-line carries both types of indices, the indices of a line must have the same value. See figure 6 where different linetypes denote different ranges of indices.

For an arbitrary graph it is not possible to distribute the indices in this manner. In this case the graph is called chequered [6, 14], i.e. the faces of the dual graph can alternately be coloured black and white such that nowhere are two black or two white faces neighbours. Such graphs emerge in the study of complex matrix models. A hint for the relation to complex matrix models is also the observation that the automorphism Θ plays the role of the complex conjugation, distinguishing diagonal and off-diagonal block matrices as real and imaginary. If the graph is not chequered the partition function vanishes, otherwise we get

$$\begin{aligned} Z &= (M^{V_1} N^{V_2} + M^{V_2} N^{V_1})(k\beta)^{P-E} \\ &= (k\beta)^{\chi} \left(\frac{MN}{k\beta}\right)^{V} (M^{-V_2} N^{-V_1} + M^{-V_1} N^{-V_2}) \end{aligned}$$

where V_1 and V_2 are the numbers of vertices whose dual plaquettes carry the same type of index, these are flip-invariants of the model, $V = V_1 + V_2$.

This model can distinguish smaller classes of graphs, the flip-move is therefore not transitive. See [6] for a discussion of the consequences of this fact.

We have seen that it is possible to classify all considered flip-invariant models and to calculate their partition functions. It is clear that these are not topological invariants of the underlying manifold, which have to be a function of the Euler characteristic, but only of the graphs with respect to the flips. In the search for topological models we have to impose additional conditions, such as the pyramid move. But the effect of such additional conditions can be understood without introducing the move: In the trivial case additional conditions will restrict the constants to fulfil $r = p\beta$. In the chequered case, a move respecting the chequered property, e.g. the Yang–Baxter move [6], will lead to $M = N$ and $Z = (k\beta)^{\chi}(\frac{M}{k\beta})^{V}$. Imposing even more conditions, which do not respect the chequered property, will force the partition function to vanish. All models claiming to be topological are therefore trivial.

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